Lecture: Convergence of Measures.

Let μ be a Radon measure on $\mathbb{R}^{n+m}$, $\mu(\mathbb{R}^{n+m}) < \infty$.

Its projection on $\mathbb{R}^n$ is

$$\sigma(A) = \mu(A \times \mathbb{R}^m), \quad A \subset \mathbb{R}^n.$$

σ is also a Radon measure.

Theorem (Slicing Measures) For σ a.e. $x \in \mathbb{R}^n$, $\exists$ p-measure Radon ν_x on $\mathbb{R}^n$ s.t.

$$\int_{\mathbb{R}^{n+m}} f \, d\mu = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f \, d\nu_x \right) d\sigma, \quad \forall f \in C_b(\mathbb{R}^{n+m}).$$

Proof. Step 1 Choose $\{f_k\} \subset C_c(\mathbb{R}^m)$ dense subset.

For each k, define

$$\gamma^k(A) = \int_{A \times \mathbb{R}^m} f_k(y) \, d\mu(x,y), \quad A \in \mathcal{B}(\mathbb{R}^n)$$

and extend it as Radon.

From

$$\gamma^k(A) \leq \|f_k\|_{L^\infty} \mu(A \times \mathbb{R}^m) = \|f_k\|_{L^\infty} \sigma(A),$$

$\gamma^k \ll \sigma$, so by R-N,

(a) $$D_\sigma \gamma^k(x) = \lim_{r \to 0} \frac{\gamma^k(\overline{B}_r(x))}{\sigma(\overline{B}_r(x))}$$ exists σ-a.e. x, σ-measurable,

and

(b) $$\int_{A \times \mathbb{R}^m} f_k(y) \, d\mu(x,y) = \int_A (D_\sigma \gamma^k) \, d\sigma, \quad \forall A \in \mathcal{B}. \quad (1)$$

Since countable union of null set is null, may assume (a) and (b) for $\forall k$.

Step 2 We have $\forall k, l$

$$D_\sigma \gamma^k(x) \leq \|f_k\|_{L^\infty} \qquad (2)$$

$$|D_\sigma \gamma^k(x) - D_\sigma \gamma^l(x)| \leq \|f_k - f_l\|_{L^\infty}. \qquad (3)$$

Pf: the first one

$$D_\sigma \gamma^k(x) = \lim_{r\to 0} \frac{\gamma^k(\overline{B}_r(x))}{\sigma(B_r(x))}$$

$$\leq \lim_{r\to 0} \frac{\int_{B_r} |f_k| \, d\mu}{\sigma(\overline{B}_r(x))}$$

$$\leq \|f_k\|_{L^\infty}.$$

the second one

$$|D_\sigma \gamma^k(x) - D_\sigma \gamma^l(x)| \leq \overline{\lim_{r\to 0}} \frac{\int_{\overline{B}_r(x)} |f_k - f_l| \, d\mu}{\sigma(\overline{B}_r(x))}$$

$$\leq \|f_k - f_l\|_{L^\infty}.$$

Step 3 For $f \in C_c(\mathbb{R}^m)$, pick $\{f_{k_j}\} \subset \{f_k\}$, $f_{k_j} \rightrightarrows f$

define $\Lambda_x(f) = \lim_{j\to\infty} D_\sigma \gamma^{k_j}(x)$.

From (3) we know that Λ_x is indept of the choice of $\{k_j\}$ and from (a) $x \mapsto \Lambda_x(f)$ is σ-measurable.

From (2),

$$|\Lambda_x(f)| \le \|f\|_{L^\infty}, \quad \forall f \in C_c(\mathbb{R}^m).$$

Riesz Rep. Thm, $\exists\, \nu_x$ s.t. $\nu_x(\mathbb{R}^m) \le \|f\|_{L^\infty}$.

$$\Lambda_x f = \int_{\mathbb{R}^m} f\, d\nu_x,$$

Inherited from (1),

$$\int_{A\times\mathbb{R}^m} f(y)\, d\mu(x,y) = \int_A \int_{\mathbb{R}^m} f(y)\, d\nu_x(y)\, d\sigma(x), \qquad f \in C_c(\mathbb{R}^m).$$

<u>Step 4</u> We write the above as

$$\int \chi_A(x) f(y)\, d\mu(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \chi_A(x) f(y)\, d\nu_x(y)\, d\sigma(x)$$

So

$$\int s(x) f(y)\, d\mu(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} s(x) f(y)\, d\nu_x(y)\, d\sigma(x)$$

for all simple fcns, By dominated conv. theorem,

$$\int g(x) f(y)\, d\mu(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} g(x) f(y)\, d\nu_x(y)\, d\sigma(x) \tag{5}$$

for all $g \in C_b(\mathbb{R}^n)$. Next, we claim this formula holds for $f \in C_b(\mathbb{R}^m)$ too. ~~For, let $f_m \to f$ uniformly on compact sets. For $\varepsilon > 0$, fix a cpt K~~ s.t.

$$\cancel{\int_{\mathbb{R}^n\times(\mathbb{R}^m\setminus K)} g(x) f(y)\, d\mu,\ \int_{\mathbb{R}^n}\int g(x) f(y)\, d\nu_x(y)\, d\sigma(x) < \varepsilon}$$

For, we may take $g, f \geq 0$ (otherwise consider $g_\pm, f_\pm$). Fix $f_n \in C_c(\mathbb{R}^n)$, $f_n \uparrow f$ and by monotone convergence theorem to get the desired conclusion.

Now, we are legal to take $f \equiv 1$,

$$\int g(x)\, d\mu(x,y) = \int_{\mathbb{R}^n} g \int_{\mathbb{R}^m} d\nu_x(y)\, d\sigma(x)$$

$$= \int_{\mathbb{R}^n} g(x)\, \nu_x(\mathbb{R}^m)\, d\sigma(x).$$

On the other hand, it is elementary to show

$$\int g(x)\, d\mu(x,y) = \int g(x)\, d\sigma(x), \quad \forall g \in C_b(\mathbb{R}^n).$$

So, $\nu_x(\mathbb{R}^m) = 1$ σ-a.e.

<u>Step 5</u> We've shown that (5) holds for $f \in C_b(\mathbb{R}^m)$, $g \in C_b(\mathbb{R}^n)$, so it also holds for $\sum f_j(y) g_j(x)$. Functions of these forms are denoted as $\mathcal{C}$. Its restriction to any cpt set forms a dense set in $C(K)$.

Now, given $f \in C_b(\mathbb{R}^n \times \mathbb{R}^m)$, we can find a sequence f_n in $\mathcal{C}$ s.t. $|f_n|_{L^\infty} \leq |f|_{L^\infty} + 1$ and $f_n \to f$ uniformly on cpt sets. Using μ, σ, ν_x all are finite, we can apply dominated convergence theorem in (5) to get

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y)\, d\mu(x,y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\, d\nu_x(y) \right) d\sigma(x) \quad \forall f \in C_b(\mathbb{R}^{n+m}).$$

<u>Theorem</u> (<u>Young Measures</u>) <u>Let</u> U be <u>open</u>, <u>bounded</u> in $\mathbb{R}^n$.
<u>For a sequence</u> $\{f_k\}_1^\infty \subset L^\infty(U;\mathbb{R}^m)$, $\|f_k\|_{L^\infty} \le C$, $\exists$ a <u>subsequence</u> $\{f_{k_j}\}$ <u>and</u> <u>Radon p-measure</u> ν_x (a.e. x) s.t.

$$F(f_{k_j}) \rightharpoonup \bar{F} \equiv \int_{\mathbb{R}^m} F(y)\, d\nu_x(y), \quad \forall F \in C(\mathbb{R}^m)$$

<u>in</u> $L^p(U)$, $\forall p > 1$.

Proof

<u>Step 1</u> $\forall A \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m)$, set

$$\mu_k(A) = \int_U \chi_A(x, f_k(x))\, d\mathcal{L}^n(x)$$

(Need to justify $x \mapsto \chi_A(x, f_k(x))$ is measurable, see [EG])

$$\mu_k(\mathbb{R}^n \times \mathbb{R}^m) = \int_U \chi_{\mathbb{R}^n \times \mathbb{R}^m}\, d\mathcal{L}^n(x) = \mathcal{L}^n(U) < \infty$$

$\exists\, \mu_{k_j} \rightharpoonup \mu$ Radon, ie

$$\int \varphi(x,y)\, d\mu_{k_j}(x,y) \longrightarrow \int \varphi(x,y)\, d\mu(x,y) \qquad \forall \varphi \in C_c(\mathbb{R}^n \times \mathbb{R}^m)$$

<u>Step 2</u> Claim :

$$\int \varphi(x,y)\, d\mu_k(x,y) = \int_U \varphi(x, f_k(x))\, d\mathcal{L}^n(x)$$

Pf : From def. of μ_k, we have

$$\int \chi_A(x,y)\, d\mu_k(x,y) = \int_U \chi_A(x, f_k(x))\, d\mathcal{L}^n(x)$$

$$\int s(x,y)\,d\mu_k(x,y) = \int_U s(x, f_k(x))\,d\mathcal{L}^n(x)$$

all simple fcns s. From $\varphi \in C_c^{\geq 0}$ can find $s_n \uparrow \varphi$ everywhere, so claim holds by monotone convergence theorem.

<u>Step 3</u> $\sigma(E) = \mathcal{L}^n(E \cap U)$, $\forall E \in \mathcal{B}(\mathbb{R}^n)$.

Recall that

$$\sigma(E) = \mu(E \times \mathbb{R}^m)$$

V open in $\mathbb{R}^n$,

$$\begin{aligned}
\sigma(V) &= \mu(V \times \mathbb{R}^m) \\
&\leq \underline{\lim}_{j\to\infty} \mu_{k_j}(V \times \mathbb{R}^m) \quad \text{(theorem 1.40 [EG])} \\
&= \underline{\lim}_{j\to\infty} \int_U \chi_{V\times\mathbb{R}^m}(x, f_{k_j}(x))\,d\mathcal{L}^n \\
&= \underline{\lim}_{j\to\infty} \int_U \chi_V(x)\,d\mathcal{L}^n(x) \\
&\leq \mathcal{L}^n(V \cap U).
\end{aligned}$$

By outer regularity,

$$\sigma(E) \leq \mathcal{L}^n(E \cap U), \quad E \in \mathcal{B}(\mathbb{R}^n).$$

Next, K cpt in $\mathbb{R}^n$, B_R a large ball so that $f_k(U) \subset B_R$ $\forall k$.

$$\begin{aligned}
\sigma(K) &= \mu(K \times \mathbb{R}^m) \\
&\geq \mu(K \times \overline{B_R}) \\
&\geq \overline{\lim}_{j\to\infty} \mu_{k_j}(K \times \overline{B_R}) \quad \text{(theorem 1.40 [EG])} \\
&= \overline{\lim}_{j\to\infty} \int_U \chi_{K\times\overline{B_R}}(x, f_{k_j}(x))\,d\mathcal{L}^n
\end{aligned}$$

$$= \overline{\lim_{j\to\infty}} \int_U \chi_K(x)\, d\mathcal{L}^n(x)$$

$$= \mathcal{L}^n(K\cap U).$$

By inner regularity,

$$\sigma(E) \geq \mathcal{L}^n(E\cap U).$$

Step 4 By slicing measures,

$$\int \varphi(x,y)\, d\mu = \int \int \varphi(x,y)\, d\nu_x(y)\, d\sigma(x)$$

$$= \int \int \varphi(x,y)\, d\nu_x(y)\, d\mathcal{L}^n(x) \quad (\text{Step } 3)$$

Combining with Step 2,

$$\int_U \varphi(x, f_{k_j}(x))\, d\mathcal{L}^n(x) \to \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} \varphi(x,y)\, d\nu_x(y)\right) d\mathcal{L}^n(x).$$

Choose $\varphi = g(x)\, F(y) \in C_c(\mathbb{R}^n \times \mathbb{R}^m)$:

$$\int_U g(x)\, F(f_{k_j}(x))\, d\mathcal{L}^n(x) \to \int_U g(x)\, \overline{F}(x)\, d\mathcal{L}^n(x) \qquad (*)$$

as $k_j \to \infty$.

For any $F\in C(\mathbb{R}^m)$, we can fix $\tilde{F} \in C_c(\mathbb{R}^m)$ s.t.

$\tilde{F} = F$ on B_R. then $\tilde{F}(f_k) = F(f_k)$ since $f_k(U) \subset B_R$.

So $(*)$ holds for $F\in C(\mathbb{R}^m)$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ in L^p-norm, $(*)$ shows that

$$F(f_{k_j}(x)) \rightharpoonup \overline{F}(x) \quad \text{in } L^p(U),\ p>1.$$

A formula on matrix product.

Let A be $n\times m$ matrix and B $m\times n$ matrix, $n\leq m$, $A(J)$, $J: j_1<\cdots<j_n$, $j_k\in\{1,2,\dots,m\}$, be the $n\times n$ matrix obtained by taking the $j_1,\dots,j_n$-th columns of A, and $B(J)$, $J: j_1<\cdots<j_n$, $j_k\in\{1,2,\dots,m\}$ be the $n\times n$ matrix obtained by taking the $j_1,\dots,j_n$-th rows of B.

We have

$$\det AB = \sum_J \det A(J) \det B(J).$$

Proof:

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j_1=1}^m a_{1j_1} b_{j_1 1} & \sum_{j_2=1}^m a_{1j_2} b_{j_2 2} & \cdots & \sum_{j_n=1}^m a_{1j_n} b_{j_n n} \\ \vdots & & & \\ \sum_{j_1=1}^m a_{nj_1} b_{j_1 1} & \sum_{j_2=1}^m a_{nj_2} b_{j_2 2} & \cdots & \sum_{j_n=1}^m a_{nj_n} b_{j_n n} \end{bmatrix}$$

$$\det(AB) = \sum_{j_1=1}^m b_{j_1 1} \det \begin{bmatrix} a_{1j_1} & \\ \vdots & * \\ a_{nj_1} & \end{bmatrix}$$

$$= \sum_{j_1, j_2} b_{j_1 1} b_{j_2 2} \det \begin{bmatrix} a_{1j_1} & a_{1j_2} & \\ \vdots & \vdots & * \\ a_{nj_1} & a_{nj_2} & \end{bmatrix}$$

$\vdots$

$$= \sum_{j_1, \cdots, j_n} b_{j_1 1} b_{j_2 2} \cdots b_{j_n n} \det \begin{bmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_n} \\ \vdots & \vdots & & \vdots \\ a_{nj_1} & a_{nj_2} & \cdots & a_{nj_n} \end{bmatrix}$$

$$\sum_{j_1, \cdots, j_n} = \sum_{1 \le i_1 < \cdots < i_n \le m} \ \sum_{\substack{\text{those} \\ \text{permutations} \\ \text{of } (i_1 \cdots i_n)}}$$

Let $j_1 \cdots j_n$ be permuted from $i_1 \cdots i_n$ by σ.

$$\det \begin{bmatrix} a_{1j_1} & \cdots & a_{1j_n} \\ \vdots & & \vdots \\ a_{nj_1} & \cdots & a_{nj_n} \end{bmatrix} = \operatorname{sgn}(\sigma) \det \begin{bmatrix} a_{1i_1} & \cdots & a_{1i_n} \\ \vdots & & \vdots \\ a_{ni_1} & & a_{ni_n} \end{bmatrix}$$

$$= \operatorname{sgn}(\sigma) \det A(I), \quad I = (i_1, \cdots, i_n)$$

$\therefore$

$$\det AB = \sum_{1 \le i_1 < \cdots < i_n \le m} \left(\sum_{\substack{\text{those} \\ \sigma}} \operatorname{sgn}(\sigma) b_{j_1 1} b_{j_2 2} \cdots b_{j_n n} \right) \det A(I)$$

$$= \sum_{1 \le i_1 < \cdots < i_n \le m} \det B(I) \det A(I), \quad \text{done. } \#$$

Special Case : Let $L : \mathbb{R}^n \to \mathbb{R}^m$ linear and use L to denote the $m \times n$ matrix of L w.r.t. canonical basis. Take $L = B$ and ~~B~~ $A = L^*$, get

$$\det(L^* L) = \sum_{1 \le j_1 < \cdots < j_n \le m} (\det L(J))^2.$$